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A modification of the Omnes method is used to solve the singular integral equations for the 3-3 partial-wave amplitudes of photoproduction. The effects of multipion production are assumed to be negligible. The method requires a knowledge of the phase at all energies. Consequently, it is necessary to treat the corresponding pion-nucleon scattering problem to determine the effect of the high-energy behavior of the phase on the solution for the scattering amplitude at low energies. The sharply resonant nature of the problen suggests an approximation in the form of solution, rather than in the Born terms, which leads to relatively simple expressions for the ratios of the 3-3 photoproduction amplitudes to the scattering amplitude and for integrals involving the 3-3 amplitudes. In addition, a modified Chew-Low formula can be derived which should satisfactorily represent the 3-3 phase shift throughout the resonance regions. Finally, the cross sections are calculated in the 3-3 approximation and the results compared with experiment.

I. INTRODUCTION

CONSIDERABLE attention has been directed toward the determination of the amplitudes for photoproduction of pions from nucleons by the technique of dispersion relations. The formulation of the dispersion relations for this process, and the first attempts to evaluate them, were made by Chew, Goldberger, Low, and Nambu[1] (hereafter referred to as CGLN). These authors obtained the integral equations for the photoproduction partial-wave amplitudes from the connection between the phases of the photoproduction and pion-nucleon scattering amplitudes provided by unitarity.[2] Only those contributions which involved the resonant 3-3 phase shift were retained under the integrals and each contributions was expanded in inverse powers of the nucleon mass M. The P-wave amplitudes generated by the nucleon total magnetic moment were determined in the static limit ($1/M \to 0$) from a comparison with the corresponding static-limit equations for the pion-nucleon scattering amplitudes.[3] In contrast, those amplitudes generated by the nucleon charge were evaluated by analogy with the cutoff model.

The various attempts to improve upon the CGLN results for the 3-3 amplitudes have met with only qualified success. These attempts invariably employ, with CGLN, the assumptions that multipion production effects may be neglected and that the 3-3 resonance exhausts the dispersion integrals. In addition to these assumptions, however, these treatments also involve either some assumption about the ratios of the photoproduction to the scattering amplitudes or some type of approximation for the inhomogeneous terms in the dispersion relations. In line with the latter approach, Solovyov and Tentiukova[4] applied the Omnes method[5]

[1] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957).
[2] K. M. Watson, Phys. Rev. **95**, 228 (1954).
[3] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).
[4] L. D. Solovyov and G. N. Tentiukova, Zh. Eksperim. i Teor. Fiz. **37**, 889 (1959) [English transl.: Soviet Phys.—JETP **10**, 634 (1960)].
[5] R. Omnes, Nuovo Cimento **8**, 316 (1958).

to the static-limit integral equations while Dennery[6] used simple pole fits to the Born term. Along different lines, McKinley[7] found an integral equation for the ratio of the magnetic-dipole amplitude generated by the magnetic moment to the scattering amplitude and solved it by numerical iteration.

In spite of the efforts of these authors, the situation with regard to the 3-3 amplitudes has not been clarified. On the one hand, there is a lack of qualitative agreement between the results of McKinley and Dennery for the energy dependence of the ratio of the magnetic dipole amplitude to the scattering amplitude. On the other hand, the 3-3 charge amplitudes found by CGLN vanish at the resonance energy, where one would ordinarily expect an enhancement. The present investigation is an attempt to resolve these uncertainties in the 3-3 photoproduction amplitudes.

The present approach differs from those of other authors in that it will be unnecessary to make approximations for the Born contributions to the integral equations. Furthermore, no assumptions will be made concerning photoproduction-to-scattering amplitude ratios, although we will derive from our solutions relatively simple expressions for these ratios. The two assumptions we shall make which have already been used by the previous authors are (1) that the equality of the 3-3 photoproduction and scattering phases, provided by unitarity at low energies, may be extended to all energies and (2) that only the 3-3 state contributes appreciably to the dispersion integrals for the amplitudes at low energy. The method to be used is a modification of the Omnes solution of singular integral equations for functions whose phase is known on the interval of singularity. Because the singular interval in the case of photoproduction extends to energy regions in which the phase is not known, it will be necessary to treat the corresponding problem for pion-nucleon scattering in order to determine the effect of the unknown "high"-energy behavior of the phase on the solution for the 3-3 scattering amplitude in the energy region from threshold through the 3-3 resonance.

II. KINEMATICS AND DISPERSION RELATIONS

Our notation is basically that of CGLN.[1] In what follows, all kinematic quantities refer to the barycentric system. In this system the differential cross section may be written

$$\frac{d\sigma}{d\Omega}=\frac{q}{k}|\langle f|\mathfrak{F}(\boldsymbol{\sigma})|i\rangle|^2, \qquad (2.1)$$

where the decomposition of the total amplitude into the usual Pauli spin matrices is given by

$$\mathfrak{F}(\boldsymbol{\sigma})=i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}\mathfrak{F}_1+\boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot(\mathbf{k}\times\boldsymbol{\varepsilon})\mathfrak{F}_2/qk +i(\boldsymbol{\sigma}\cdot\mathbf{k})\mathbf{q}\cdot\boldsymbol{\varepsilon}\mathfrak{F}_3/qk+i(\boldsymbol{\sigma}\cdot\mathbf{q})\mathbf{q}\cdot\boldsymbol{\varepsilon}\mathfrak{F}_4/q^2. \qquad (2.2)$$

In these expressions, $\boldsymbol{\varepsilon}$ is the photon polarization vector and q and k are the barycentric momenta of the pion and photon, respectively. The magnitudes of these momenta, together with the pion energy ω_q and the initial and final nucleon energies E_1 and E are related to the total barycentric energy W by the expressions[8]

$$k=(W^2-M^2)/2W,\; \omega_q=k+1/2W,\; q=(\omega_q^2-1)^{1/2}, \qquad (2.3)$$
$$E_1=W-k,\quad E=W-\omega_q,$$

where M is the nucleon mass. If $\mathfrak{F}$ is decomposed into linearly independent isotropic matrices according to

$$\mathfrak{F}=\mathfrak{F}^+\delta_{\beta 3}+\mathfrak{F}^-\tfrac{1}{2}[\tau_\beta,\tau_3]+\mathfrak{F}^0\tau_\beta,$$

then the connection between the $\mathfrak{F}^\alpha$ ($\alpha=+,-,0$) and the amplitudes of the four possible charge configurations are given by

$$\mathfrak{F}\begin{Bmatrix}\gamma p\to\pi^0 p\\ \gamma n\to\pi^0 n\end{Bmatrix}=\mathfrak{F}^+\pm\mathfrak{F}^0,$$
$$\mathfrak{F}\begin{Bmatrix}\gamma p\to\pi^+ n\\ \gamma n\to\pi^- p\end{Bmatrix}=\sqrt{2}(\mathfrak{F}^0\pm\mathfrak{F}^-). \qquad (2.4)$$

To make use of the unitarity condition, one must decompose the photoproduction amplitudes into photon multipole eigenamplitudes which correspond to transitions into eigenstates of the final pion-nucleon system with a definite total angular momentum J, isotopic spin T, and parity. The correspondence between the isospin $(+,-)$ amplitudes and the eigenamplitudes with eigenvalues $T=\frac{3}{2}$ and $T=\frac{1}{2}$ is given by

$$\mathfrak{F}^{3/2}=\mathfrak{F}^+-\mathfrak{F}^-,\quad \mathfrak{F}^{1/2}=\mathfrak{F}^++2\mathfrak{F}^-, \qquad (2.5)$$

while the amplitude $\mathfrak{F}^0$ corresponds only to the value $T=\frac{1}{2}$. The complete angular momentum decomposition of $\mathfrak{F}$ into photon-multipole eigenamplitudes was given by CGLN. Here, however, we are concerned only with the $J=\frac{3}{2}$, even-parity part of the amplitude $\mathfrak{F}^{3/2}$, defined by[9]

$$\mathfrak{F}^{33}=\frac{1}{4\pi}\int\frac{d\Omega_{q'}}{qq'}[3\mathbf{q}\cdot\mathbf{q}'-(\boldsymbol{\sigma}\cdot\mathbf{q})\boldsymbol{\sigma}\cdot\mathbf{q}']\mathfrak{F}^{3/2}(\mathbf{q}',\mathbf{k}). \qquad (2.6)$$

The amplitude $\mathfrak{F}^{33}$ may be expressed in terms of the CGLN multipole amplitudes $M_{1+}{}^{3/2}$ and $E_{1+}{}^{3/2}$, which correspond to transitions induced by magnetic-dipole and electric-quadrupole radiation, respectively. We will find it convenient to deal not with these multipole amplitudes but with the linear combinations

$$\phi_1=(M_{1+}{}^{3/2}-E_{1+}{}^{3/2})/qh,$$
$$\phi_2=(M_{1+}{}^{3/2}+E_{1+}{}^{3/2})/qhk, \qquad (2.7)$$

[6] P. Dennery, Phys. Rev. **124**, 2000 (1961).

[7] J. M. McKinley, University of Illinois Tech. Report No. 38 1962 (unpublished). Part of this work is reproduced in Rev. Mod. Phys. **35**, 788 (1963).

[8] We have set $\hbar=c=1$ and all energies are expressed in units of the pion mass.

[9] S. Gartenhaus and R. Blankenbecler, Phys. Rev. **116**, 1350 (1959); **116**, 1297 (1959).

where

$$h(W)=(W-M)(E_1+M)^{1/2}(E+M)^{1/2}/2W. \quad (2.8)$$

In terms of these amplitudes the parts of $\mathfrak{F}^{3/2}$ which, by unitarity, have the 3-3 phase are given by

$$\begin{aligned}
\mathfrak{F}_1^{33}&=3\cos\theta(M_{1+}^{3/2}+E_{1+}^{3/2})=3qhk\phi_2\cos\theta\,,\\
\mathfrak{F}_2^{33}&=2M_{1+}^{3/2}=qh(k\phi_2+\phi_1)\,,\\
\mathfrak{F}_3^{33}&=-3(M_{1+}^{3/2}-E_{1+}^{3/2})=-3qh\phi_1\,,\\
\mathfrak{F}_4^{33}&=0.
\end{aligned} \quad (2.9)$$

The amplitudes $\mathfrak{F}_i$ ($i=1, 2, 3, 4$) are linear combinations of four crossing-symmetric invariant amplitudes which satisfy the fixed-momentum-transfer dispersion relations given in CGLN. These amplitudes have singularities corresponding to (1) single-particle exchange—the so-called Born terms, (2) exchange of two (or more) particles in the physical region, and (3) the analogous two-particle singularities demanded by crossing symmetry—which we call the left-hand-cut terms. By employing the projection operator defined in Eq. (2.6), one can obtain the dispersion relations satisfied by the $\phi_i(i=1, 2)$.

If we retain only the 3-3 amplitudes under the dispersion integrals, we find the expressions ($\omega=W-M$)

$$\phi_i=\phi_{iI}+\frac{1}{\pi}\int_1^\infty d\omega'\frac{\mathrm{Im}\phi_i(\omega')}{\omega'-\omega}, \quad (2.10\text{a})$$

where the inhomogeneous terms ϕ_{iI} can be written

$$\phi_{1I}=\phi_{1B}+\phi_{1L},\quad \phi_{2I}=\phi_{2B}+\phi_{2L}+\beta/\omega. \quad (2.10\text{b})$$

In these expressions, the ϕ_{iB} are the Born projections [see Eq. (A1) of the Appendix], the ϕ_{iL} are the 3-3 projections from the left-hand cut, and β is given by

$$\beta=\frac{1}{\pi}\int_1^\infty\frac{d\omega'}{\omega'}\,\mathrm{Im}\phi_1(\omega'). \quad (2.10\text{c})$$

The left-hand-cut contributions are typically less than 5% of the full amplitude. It is therefore consistent with our approach, insofar as we have already neglected the contributions to these integrals from other states, to retain only the static limit of these terms. The results of a $1/M$ expansion for the ϕ_{iL}, in the limit $M\rightarrow\infty$, are given by

$$\begin{aligned}
\phi_{iL}&=\frac{1}{9\pi}\int_1^\infty d\omega'\frac{\mathrm{Im}(2k'\phi_2'-\phi_1')}{\omega'+\omega},\\
\phi_{2L}&=\frac{1}{9\pi}\int_1^\infty d\omega'\left[\frac{\mathrm{Im}(\phi_2'-2\phi_1'/k')}{\omega'+\omega}+\frac{\mathrm{Im}(2\phi_2'-\phi_1'/k')}{\omega}\right].
\end{aligned} \quad (2.11)$$

Omnes[5] has shown that if the phase δ of ϕ_i is known in the interval $(1, \infty)$, so that the substitution

$$\mathrm{Im}\phi_i=\phi_i e^{-i\delta}\sin\delta$$

can be made in Eq. (2.10), then the solution to the resulting integral equation can be written

$$\phi_i(\omega)=\phi_{iI}(\omega)+e^{\Delta(\omega)}\frac{1}{\pi}\int_1^\infty d\omega' e^{-\rho(\omega')}\frac{\sin\delta(\omega')}{\omega'-\omega}\phi_{iI}(\omega')\,,$$

where $\quad (2.12)$

$$\Delta(\omega)=\frac{1}{\pi}\int_1^\infty\frac{d\omega'\delta(\omega')}{\omega'-\omega}$$

and where, for real $\omega\geq 1$,

$$\rho(\omega)=\lim_{\epsilon\rightarrow 0}\Delta(\omega+i\epsilon)-i\delta.$$

This solution is valid, according to Omnes, if δ is continuous and if $\delta(\infty)=0$. Alternatively, if the inhomogeneous term ϕ_{iI} has the integral representation

$$\phi_{iI}(\omega)=\frac{1}{2\pi i}\int_{C_I}d\omega'\frac{\phi_{iI}(\omega')}{\omega'-\omega},$$

where the contour C_I encloses only the singularities of ϕ_{iI}, then the integral equation has the solution[10]

$$\phi_i(\omega)=e^{\Delta(\omega)}\frac{1}{2\pi i}\int_{C_I}d\omega' e^{-\Delta(\omega')}\frac{\phi_{iI}(\omega')}{\omega'-\omega}. \quad (2.13)$$

This form of the solution is valid whenever Eq. (2.12) hold and also in some cases where the phase is discontinuous.

In order to apply Eq. (2.13) to the photoproduction amplitudes, one must know the phase at all physical energies. This information is available from phase-shift analyses of pion-nucleon scattering only up to 600 MeV (pion lab energy).[11] If we assume, however, that the Omnes method yields physically meaningful solutions, then we may hope to learn something of the unknown portion of the phase from a consideration of the scattering amplitude, which satisfies a dispersion relation similar to those for the photoproduction amplitudes. Our rationale is the following: If we can use the Omnes method to "solve" the scattering equation, i.e., to determine the function $\Delta(\omega)$, then, by unitarity, we can use the same Δ to evaluate the photoproduction amplitudes. Our plan therefore is to construct that part of Δ which is generated by the unknown, high-energy behavior of the phase in such a way that the solution reproduces the scattering amplitude at low energies, where it is known.

III. THE 3-3 SCATTERING AMPLITUDE AND PHASE

The dispersion relations satisfied by the 3-3 scattering amplitudes may be obtained by projection from the

[10] W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1609 (1960).

[11] Recent phase-shift analyses indicated that there is no inelasticity in the 3-3 state below $T_\pi=700$ MeV. Thus, at least up to this energy, the 3-3 scattering and photoproduction phases are equal.

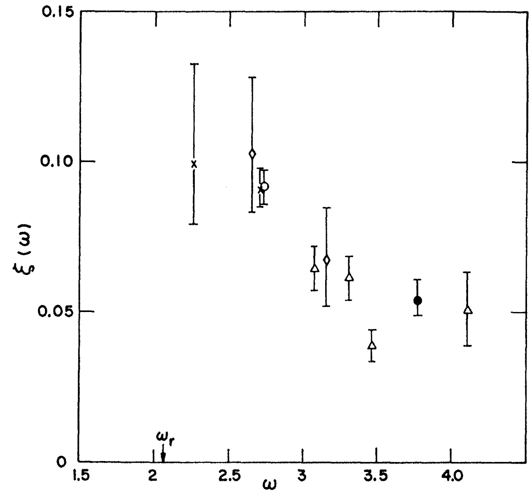

FIG. 1. Plot of $\xi(\omega)=\omega(\omega_r-\omega)/(\omega_r q^3 \cot\delta_{33})$ versus ω. For the resonance energy we have used $\omega_r=2.07$. The experimental points are those of Table I.

fixed-momentum-transfer dispersion relations for the invariant amplitudes [see Eqs. (3.3) and (3.4) of Ref. 12]. In line with our treatment of the photoproduction amplitudes, we retain only the 3-3 amplitude under the dispersion integrals and we use the static-limit of the small left-hand-cut terms. The result of this projection is that the amplitude ψ, defined by

$$\psi=\frac{2W}{(E+M)}\frac{f_{1+}{}^{3/2}}{q^2}=\frac{2W}{E+M}\frac{e^{i\delta_{33}}\sin\delta_{33}}{q^3} \tag{3.1}$$

has the integral representation

$$\psi(\omega)=\psi_I(\omega)+\frac{1}{\pi}\int_1^\infty d\omega'\frac{\mathrm{Im}\psi(\omega')}{\omega'-\omega}, \tag{3.2a}$$

where

$$\psi_I=\psi_B+\psi_L. \tag{3.2b}$$

The small term ψ_L is given by

$$\psi_L=\frac{1}{9\pi}\int_1^\infty d\omega'\frac{\mathrm{Im}\psi(\omega')}{\omega'+\omega}, \tag{3.2c}$$

while the Born projection may be found in Eq. (A2) of the Appendix. If δ is the phase of ψ, then Eq. (3.2) has a solution similar to that for the photoproduction amplitudes, namely

$$\psi(\omega)=e^{\Delta(\omega)}\frac{1}{2\pi i}\int_{C_I'}\frac{d\omega' e^{-\Delta(\omega')}\psi_I(\omega')}{\omega'-\omega}, \tag{3.3}$$

where the contour C_I' encloses the singularities of the inhomogeneous term ψ_I.

The 3-3 phase shift δ_{33} which appears in Eq. (3.1) is well represented below resonance by the Chew-Low formula[3]

$$q^3\cot\delta_{33}=3\omega(\omega_r-\omega)/(4f^2\omega_r),\quad \omega_r\geq\omega, \tag{3.4a}$$

where ω_r is the resonance position and f is the pion-nucleon coupling constant. We will use the values $\omega_r=2.07$ and $f^2=0.082$.

TABLE I. 3-3 phase shift above resonance.

Lab pion energy (MeV)		(Phase shift)δ_{33} (degrees)
220	2.27	111.5 ± 5[a]
294	2.66	128 ± 6[b]
307	2.72	132.6 ± 2[a]
310	2.74	133.2 ± 1.7[c]
370	3.04	147.7 ± 3[d]
395	3.16	147 ± 6[b]
430	3.33	150.6 ± 3[d]
460	3.47	160 ± 2[d]
525	3.78	155.3 ± 2.5[e]
600	4.11	157 ± 5[d]

[a] H. Y. Chiu and E. L. Lomon, Ann. Phys. (N. Y.) 6, 50 (1959).
[b] H. L. Anderson and W. C. Davidson, Nuovo Cimento 5, 1238 (1957).
[c] H. H. Foote, O. Chamberlain, E. H. Rogers, H. M. Steiner, C. Weigand, and T. Ypsilantis, Phys. Rev. Letters 4, 30 (1960).
[d] W. D. Walker, J. Davis, and W. D. Shepard, Phys. Rev. 118, 1612 (1960).
[e] M. E. Blevins, M. M. Block, and J. Leitner, Phys. Rev. 112, 1287 (1958).

Some experimental values of the phase above resonance are shown in Table I, where one can see that δ_{33} tends to approach π. To illustrate this tendency more clearly, we have plotted the function $\xi(\omega)=\omega(\omega_r-\omega)/(\omega_r q^3\cot\delta_{33})$ in Fig. 1. Instead of having the constant value $4f^2/3$ predicted by Eq. (3.4), the values of ξ computed from the phases of Table I decrease rapidly toward zero.[13] A zero in $\xi(\omega)$ somewhere in the interval $5\leq\omega\leq10$ is seen to be consistent with the data.

Two of the simplest assumptions one can make concerning the very high-energy behavior of the phase shift are (1) that δ_{33} passes through π at some point ω_m and subsequently approaches π from above, and (2) that δ_{33} turns over and approaches zero from above. These assumptions, however, have consequences which differ little from one another—at least insofar as they affect the construction of the Omnes solution for the scattering amplitude in the low-energy region. That this is so follows from the fact that there is an important distinction between the phase δ of the elastic scattering amplitude ψ and the corresponding phase shift δ_{33}. The distinction is important because it is the amplitude phase which must be used in the Omnes method. While the only condition on the phase shift is that of continuity, from Eq. (3.1) we have $\mathrm{Im}\psi\geq0$—that is, the amplitude phase must satisfy the requirement $0\leq\delta\leq\pi(\mathrm{mod}2\pi)$. Thus, whenever δ_{33} passes through some multiple of π, δ must have a discontinuity. One can readily show that if δ has the discontinuity $-\pi$ at ω_m, then the function e^Δ and, therefore, the solution for ψ,

[12] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. 106, 1337 (1957).

[13] The tendencey of $\xi(\omega)$ to approach zero persists even in the more recent phase-shift analyses. See, for example, P. Auvil, A. Donnachie, A. T. Lea, and C. Lovelace, Phys. Letters 10, 132 (1964).

have the expected zero at ω_m. It follows that under either assumption (1) or (2) above, the amplitude phase rises through $\pi/2$ at ω_r and decreases through $\pi/2$ at some ω_m. This behavior is illustrated in Fig. 2. Consequently, the Omnes solutions for the two cases differ appreciably only in the vicinity of ω_m.

On the basis of these arguments we can adopt the simpler assumption (1) and represent the phase above resonance by the expression

$$\delta_{33}=\pi\left[1-\frac{1}{2}\left(\frac{\omega_m-\omega}{\omega_m-\omega_r}\right)\left(\frac{\omega_r-\alpha}{\omega-\alpha}\right)\right],\quad \omega_r\leq\omega\leq\omega_m\,, \tag{3.4b}$$

$$\delta=0\,,\quad \omega_m\leq\omega\,,$$

where α is to be found from the condition $\delta(2.74)=0.74\pi$, in agreement with the 310-MeV phase shift of Table I. The phase given by Eq. (3.4), which is to be used as input in the Omnes solution, agrees with the experimental values for δ_{33} throughout the region where they are known.

It is not to be expected that our assumption correctly describes the behavior of the phase at high energies, where inelasticity is certain to play a role, but it should be sufficient to account for the effect of that behavior on the Omnes solution at low energies.

We now turn to the determination of ω_m. If the 3-3 phase were known for all energies, then it should be possible to write down an Omnes solution for the scattering amplitude itself, as in Eq. (3.3). This solution would automatically have the correct phase and, if the Omnes method is correct, it would also satisfy the unitarity condition on the scattering amplitude. In the present case, where the high-energy behavior of the phase is not known, we can construct solutions using the known phase at low energies. By varying the high-energy behavior of the phase, we would arrive at a family of solutions which have the correct phase at low energy, but not all of which satisfy unitarity there. If our assumption concerning the high-energy behavior of the phase is reasonably correct, then one of these solutions will satisfy unitarity reasonably well at low energies. In particular, one can always find a solution which satisfies unitarity exactly at some particular energy. The natural choice in the problem at hand is the resonance energy. At resonance the scattering amplitude satisfies the condition

$$|\psi(\omega_r)|=2W_rq_r^{-3}(E_r+M)^{-1} \tag{3.5}$$

and we will require our solution for $\psi(\omega)$ to satisfy this condition. With our representation for the phase above resonance, this condition leads to a unique value for ω_m.

In general, the evaluation of solutions having the form of Eq. (3.3) involves the performance of a complicated integral around the contour C_I' after a detailed analysis of the singularity structure of the inhomogeneous term ψ_I. Here, however, it is possible to circumvent both of these complications. This follows

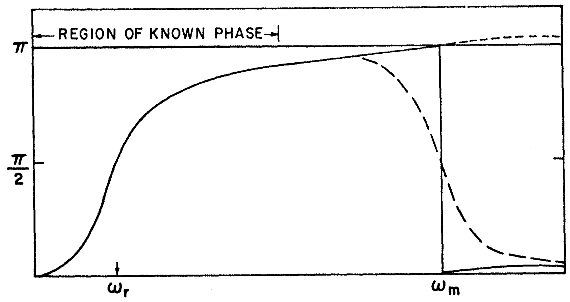

FIG. 2. Behavior of the amplitude phase above the region of known phase. The short-dashed curve represents the extension of the phase shift under the assumption that it passes through π at ω_m, while the solid curve gives the corresponding amplitude phase. The dashed curve is the phase in the case where the phase shift drops rapidly toward zero.

from the fact that the phase δ rises rapidly through $\pi/2$ at ω_r and, according to our assumption, falls rapidly from π to 0 at ω_m. Thus, for the purpose of calculating $\Delta(\omega)$ away from the physical region we can take $\delta\approx\pi$ for $\omega_r\leq\omega\leq\omega_m$ and $\delta\approx0$ elsewhere in the defining integral. This approximation leads to the result

$$e^{-\Delta(\omega')}=(\omega_m-\omega')/(\omega_r-\omega') \tag{3.6}$$

for ω' sufficiently far from the physical region. This approximate form was compared with $e^{-\Delta}$ computed exactly from the phase in Eq. (3.2). It was found that the error is $\lesssim 3\%$ throughout the region of the singularities of ψ_I; the maximum error occurred at $\omega=0$, the position of the singularity nearest the physical region. Therefore, to the accuracy indicated, we can use this approximation under the integrals in the Omnes solution.

When $e^{-\Delta(\omega')}$ from Eq. (3.6) is substituted into Eq. (3.3), the only integrals which remain can be identified with the Cauchy-integral representations for the inhomogeneous term ψ_I. The expression which results for ψ is

$$\psi(\omega)\equiv\frac{2W}{E+M}\frac{e^{i\delta_{33}}\sin\delta_{33}}{q^3}$$

$$=\frac{e^{\Delta(\omega)}}{\omega_m-\omega}[(\omega_r-\omega)\psi_I(\omega)+(\omega_m-\omega_r)\psi_I(\omega_m)]. \tag{3.7}$$

While this approximation is relativistic in the sense that the inhomogeneous terms may be treated exactly, we cannot expect it to be correct at very high energies where the phase is unknown. The principal advantage of Eq. (3.7) is, of course, that the solution in the physical region involves the inhomogeneous term only in the physical region.

Another advantage presented to us by Eq. (3.7) is that certain nonsingular integrals involving the 3-3 amplitudes, such as the one which appears in Eq. (3.2c), can be evaluated in closed form. For example,

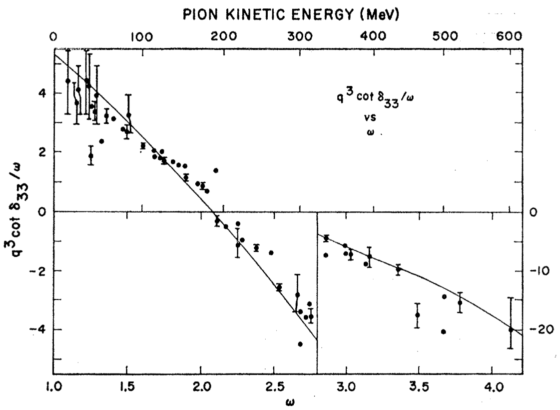

FIG. 3. Chew-Low Plot: $q^3\cot\delta_{33}/\omega$ versus ω. The solid curve is the prediction of Eq. (3.13) for the values $\omega_r=2.07$, $\omega_m=6.38$, and $f^2=0.082$. The experimental points are from the compilation of McKinley (Ref. 7).

one can readily obtain the approximate formula

$$\frac{1}{\pi}\int_1^\infty d\omega'\frac{\operatorname{Im}\psi(\omega')}{\omega'-\sigma}\equiv\psi(\sigma)-\psi_I(\sigma)=\frac{\omega_m-\omega_r}{\omega_r-\sigma}\psi_I(\omega_m)\,, \quad (3.8)$$

where σ is not in the physical region. This result follows immediately when one substitutes Eq. (3.6) for $e^{\Delta(\sigma)}$ into the above solution for $\psi(\sigma)$. Similarly, if G is any function analytic on the physical cut $(1,\infty)$, then from Eq. (3.8) and the Cauchy-integral representation for G one can obtain the result

$$\frac{1}{\pi}\int_1^\infty d\omega' G(\omega')\operatorname{Im}\psi(\omega')=(\omega_m-\omega_r)\psi_I(\psi_m)G(\omega_r)\,. \quad (3.9)$$

The appearance of $G(\omega_r)$ in this result clearly indicates the sharp-resonance nature of our approximation.

It is now a simple matter to determine the parameter ω_m. The inhomogeneous term ψ_I is just the sum of the Born term and the left-cut term. We can use Eq. (3.8) to put ψ_L in the form

$$\psi_L(\omega)=-\frac{1}{9}\frac{(\omega_m-\omega_r)}{(\omega+\omega_r)}[\psi_B(\omega_m)+\psi_L(\omega_m)]\,.$$

The normalization condition (3.5), when applied to the expression (3.7) for ψ, assumes the form

$$e^{-\rho(\omega_r)}=\psi_I(\omega_m)(E_r+M)q_r^3/(2W_r)\,, \quad (3.10)$$

where

$$\psi_I(\omega_m)\equiv\psi_B(\omega_m)+\psi_L(\omega_m)=\frac{\psi_B(\omega_m)}{1-\frac{1}{9}(\omega_m-\omega_r)/(\omega_m+\omega_r)}$$

and where the principal-value integral for $\rho(\omega_r)$ is to be computed with the phase δ in Eq. (3.4). Equation (3.10) was solved graphically for ω_m. With the resonance position at $\omega_r=2.07$ and the coupling constant given by $f^2=0.082$, we found the result $\omega_m=6.38$. In Table II

TABLE II. Results for the 3-3 amplitudes.[a]

Amplitude	Left-cut contribution	β (Residue of pole at $\omega=0$)	Resultant inhomogeneous term at ω_m
$\psi\Big/\left(\frac{g^2}{3M^2}\right)$	$\frac{0.1090}{\omega_r+\omega}$	$\cdots$	0.2262
$\phi_1{}^\mu\Big/\left(\frac{g(\mu_p-\mu_n)}{3M}\right)$	$\frac{0.0820}{\omega_r+\omega}$	$\cdots$	0.1848
$\phi_2{}^\mu\Big/\left(\frac{g(\mu_p-\mu_n)}{3M}\right)$	$\left(\frac{-0.0400}{\omega_r+\omega}+\frac{0.0479}{\omega}\right)$	0.3845	0.0973
$\phi_1{}^e\Big/\left(\frac{eg}{6M}\right)$	$\frac{0.0054}{\omega_r+\omega}$	$\cdots$	0.0242
$\phi_2{}^e\Big/\left(\frac{eg}{6M}\right)$	$\left(\frac{-0.0066}{\omega_r+\omega}+\frac{0.0037}{\omega}\right)$	0.0505	0.0097

[a] Quoted results are for the case $f^2=0.082$, $\omega_r=2.07$, and $\omega_m=6.38$.

we list our results for the left-cut term and the total inhomogeneous term at ω_m.

We can put our results in the form of a generalized Chew-Low formula. If we define a function $\eta(\omega)$ according to

$$\operatorname{Re}e^{-\Delta(\omega)}\equiv e^{-\rho}\cos\delta=\eta(\omega)(\omega_r-\omega)/(\omega_m-\omega)\,,\qquad 1\leq\omega\leq\omega_m\,, \quad (3.11)$$

then we expect η to be smooth and slowly varying in the physical region. Furthermore, inasmuch as Eq. (3.6) is valid for $\omega\leq 1$, we expect to have $\eta\simeq 1$. When we use the phase of Eq. (3.4) in the computation of η we find that throughout the region $1\leq\omega\leq\omega_m$, $\eta(\omega)$ can be represented to 2% by the non-unique form

$$\eta(\omega)=1+(\omega_r-\omega)/(2M)\,. \quad (3.12)$$

Now that η is known, its definition can be used to eliminate e^ρ from Eq. (3.7); we thus find the formula

$$q^3\cot\delta_{33}=\left(\frac{2W}{E+M}\right)\frac{(\omega_r-\omega)\eta(\omega)}{(\omega_r-\omega)\psi_I(\omega)+(\omega_m-\omega_r)\psi_I(\omega_m)}\,. \quad (3.13)$$

One can regain the usual Chew-Low formula if one first goes to the static limit, where $\psi_I=4f^2/(3\omega)$, and then lets ω_m approach infinity. The result (3.13) is compared with the experimental phases in Fig. 3.

Although we have required an approximate knowledge of the 3-3 phase in order to arrive at the result (3.13), it may yet be useful in some future analysis of the scattering data. This is so because the approximations used to derive it are roughly independent of the precise values of ω_r and ω_m. Thus, Eq. (3.13) may be fit to the experimental data as a three-parameter formula. However, because of the fact that from the present point of view one of the parameters—the zero position ω_m—is related to the other two by the normalization procedure, a more profitable approach would be to treat the combined expressions (3.10) and (3.13) as a two-parameter representation of δ_{33} in terms of ω_r and f^2. In this way one might obtain not only a satisfactory representation for the phase shift but also a more precise determination of the resonance position and coupling constant.

IV. THE 3-3 PHOTOPRODUCTION AMPLITUDES

Now that ω_m and the function e^{Δ} have been found, we can proceed to determine the photoproduction amplitudes ϕ_1, ϕ_2 defined in Eq. (2.7). The Omnes solution for these amplitudes is given by Eq. (2.13). As in the case for the scattering amplitude, the approximation (3.6) can be used to help perform the contour integral. The photoproduction amplitudes then take the form

$$\phi_i=e^{\Delta(\omega)}\left[\frac{(\omega_r-\omega)\phi_{iI}(\omega)+(\omega_m-\omega_r)\phi_{iI}(\omega_m)}{\omega_m-\omega}\right]. \quad (4.1)$$

As before, all nonsingular integrals involving these amplitudes can be carried out with the help of Eqs. (3.8–3.9) with ψ replaced by ϕ_i. In particular, these equations can be used to determine β and the $\phi_{iL}(\omega)$ from Eqs. (2.10c), and (2.11). When the resulting expressions are inserted into Eq. (2.10b) for the $\phi_{iL}(\omega_m)$, there remains a pair of coupled linear equations for the $\phi_{iI}(\omega_m)$ in terms of the known $\phi_{iB}(\omega_m)$. Given ω_m and the resonance position ω_r, the solution of these coupled equations is straightforward. Our results, determined with $\omega_m=6.38$ and $\omega_r=2.07$, are compiled in Table II.

While solution (4.1) for the ϕ_i is satisfactory as it stands, it is more instructive and more convenient in practice to work with the ratios of the photoproduction amplitudes to the scattering amplitude. We can readily construct these ratios from Eqs. (3.13) and (4.1) and we find the result

$$\frac{\phi_i}{\psi}=\frac{\phi_i}{[2W/(E+M)]e^{i\delta_{33}}\sin\delta_{33}}$$
$$=\frac{(\omega_r-\omega)\phi_{iI}(\omega)+(\omega_m-\omega_r)\phi_{iI}(\omega_m)}{(\omega_r-\omega)\psi_I(\omega)+(\omega_m-\omega_r)\psi_I(\omega_m)}. \quad (4.2)$$

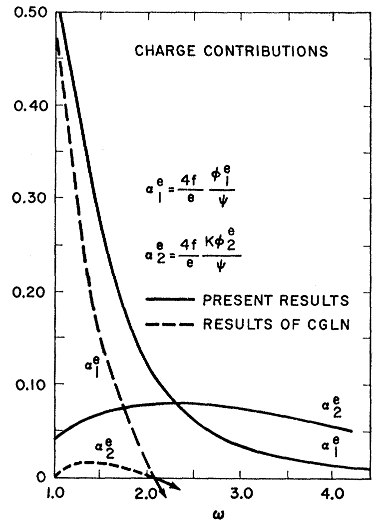

FIG. 5. Ratios of the photoproduction amplitudes generated by the nucleon charge to the scattering amplitude. The solid curves are the predictions of Eq. (4.2). The corresponding predictions of CGLN (dashed curves) are shown only up to the resonance position, where they vanish linearly.

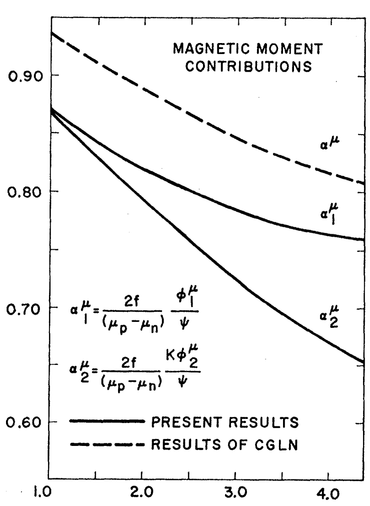

FIG. 4. Ratios of the photoproduction amplitudes generated by the total nucleon magnetic moment to the scattering amplitude. The solid curves are the predictions of Eq. (4.2). The CGLN results, which predict $\alpha_1^\mu=\alpha_2^\mu$, are given by the dashed curve.

The advantage of a closed form such as (4.2) for the amplitude ratios is that the dependence of these ratios on the parameters ω_r, ω_m, and the coupling constant is made explicit. The ratios predicted by Eq. (4.2) for the amplitudes generated by the nucleon total magnetic moment (μ) and charge (e) are shown in Figs. 4 and 5, respectively. Also shown in these figures are the corresponding predictions of CGLN[1]; however, the ratios for the CGLN charge terms are shown only up to ω_r, where they change sign.

It can be shown that the CGLN prescription for the determination of the 3-3 charge amplitudes is a special case of the solution (4.1). This solution may be rewritten in the form

$$\phi_i{}^e=e^{i\delta}\cos\delta\phi_{iI}{}^e(\omega)+e^{i\delta}\left(\frac{\cos\delta}{\omega_r-\omega}\right)(\omega_m-\omega_r)\phi_{iI}{}^e(\omega_m), \quad (4.3)$$

where we have used Eq. (3.11) with $\eta(\omega)=1$. The second term in the last expression is finite at resonance because $\cos\delta$ vanishes there. In the limit $\omega_m\rightarrow\infty$,

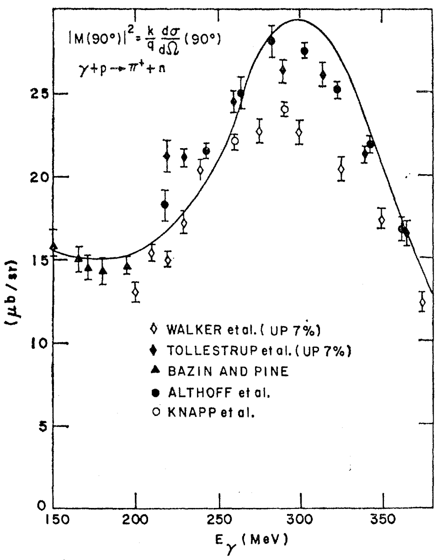

FIG. 6. Matrix element squared at 90° for π^+ photoproduction. The present predictions are compared with various experiments. References 15–19.

Eq. (4.3) reduces to the expression

$$\phi_i^e = e^{i\delta}\cos\delta\phi_{iB}^e; \quad i=1,2. \tag{4.4}$$

This result follows from the fact that the Born terms $\phi_{iB}^e(\omega)$ approach zero faster than $1/\omega$ as $\omega\rightarrow\infty$; consequently, all of the non-Born inhomogeneous terms evaluated by the procedure outlined at the beginning of the section vanish in the limit $\omega_m\rightarrow\infty$. The result (4.4), however, is just the CGLN expression for the charge terms and the difference between expressions (4.3) and (4.4) essentially represents the present corrections to these terms.

A comparison of our results with the photoproduction cross sections can be made. It should be noted, however, that such a comparison should be made only after reliable results have been obtained for the amplitudes connected with phase shifts other than δ_{33}. Because of the lack of information concerning these other amplitudes, we adopt a procedure similar to that of Ball,[14] who retained only the 3-3 amplitudes under the integrals in the CGLN dispersion relations. Our procedure, entirely equivalent to the one above, is as follows. From the integral representation for the full amplitude $\mathfrak{F}$, we project out the 3-3 part $\mathfrak{F}^{33}=p^{33}\mathfrak{F}$, so that we have for the full amplitude

$$\mathfrak{F}=\mathfrak{F}^{33}+(1-p^{33})\mathfrak{F}.$$

For $\mathfrak{F}^{33}$ we use Eq. (2.9) together with our solutions for ϕ_1 and ϕ_2. The remaining term, $(1-p^{33})\mathfrak{F}$, involves only nonsingular integrals and so we use Eq. (3.9) in evaluating them.

Our results for the photoproduction cross sections are compared with experiment in Figs. 6–10. We reiterate that these calculations are meaningful only insofar as the so-called "small phases" can be set equal to zero. With this limitation in mind we see that our results are in quite good agreement with the data. However, the fairly large discrepancies among the results of the various experimental groups prevent us from drawing any conclusions about the accuracy of our predictions for the 3-3 amplitudes. For the case of charged-pion photoproduction, we show the matrix element squared at 90° (Fig. 6) and the differential cross section at 260 MeV (Fig. 7). It can be seen that the theoretical curves tend to agree best with the higher data in the vicinity of resonance.

The cross sections for the process $\gamma p\rightarrow\pi^0 p$ are reported in terms of the coefficients in the expansion

$$\frac{d\sigma^{\pi 0}}{d\Omega}=A+B\cos\theta+C\cos^2\theta+\cdots.$$

These coefficients are shown in Figs. 8–10. The predictions for the coefficient A in Fig. 8 appear to be too large near resonance.

V. DISCUSSION AND CONCLUSIONS

Our primary aim has been to improve the calculation of the 3-3 photoproduction amplitudes from the CGLN dispersion relations. Except for a slight modification, the method used is that of the Omnes. To carry out this program we have made three basic assumptions, each of which compensates for some aspect of our present lack of knowledge of the photoproduction amplitudes. These assumptions are (1) that only the 3-3 amplitudes contribute appreciably to the dispersion integrals at energies below and in the vicinity of the 3-3 resonance, (2) that the phase of the 3-3 photoproduction amplitudes is the same as that of the 3-3 scattering amplitude for all physical energies, and (3) that the effect of the unknown high-energy behavior of the phase on the solution for the 3-3 scattering amplitude can be represented by a zero in that amplitude at some energy ω_m in the physical region. The parameter ω_m in the last assumption is not arbitrary but is determined from the unitarity condition at resonance.

The principal result of this investigation is contained in the expression (4.2) for the ratios of the 3-3 photoproduction amplitudes to the scattering amplitude.

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The present predictions for these ratios are compared with those of CGLN in Figs. 4 and 5. The significant features that one observes from this comparison are (1)

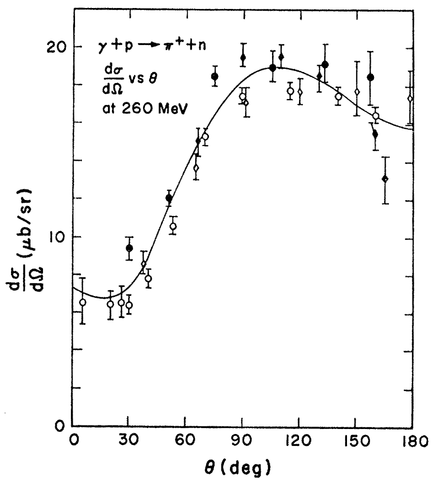

FIG. 7. Barycentric differential cross section at 260 MeV. The data notation is the same as in Fig. 6.

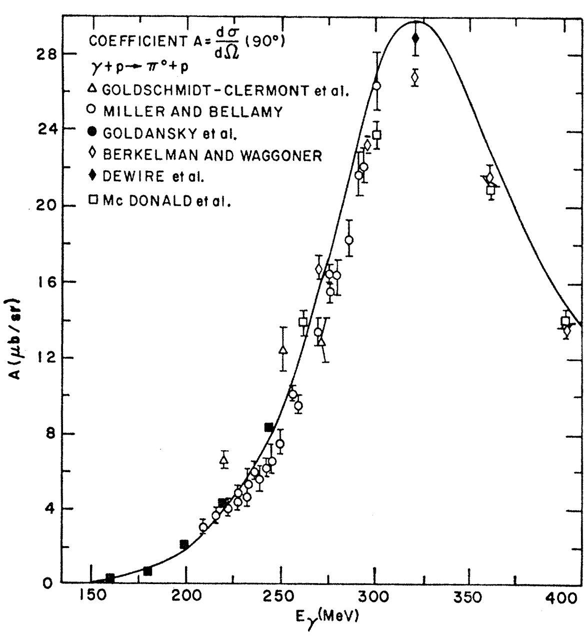

FIG. 8. The coefficient A in the expansion $d\sigma/d\Omega^{\pi^0}=A+B\cos\theta+C\cos^2\theta+\cdots$ for the process $\gamma p\rightarrow\pi^0 p$. References 20–25.

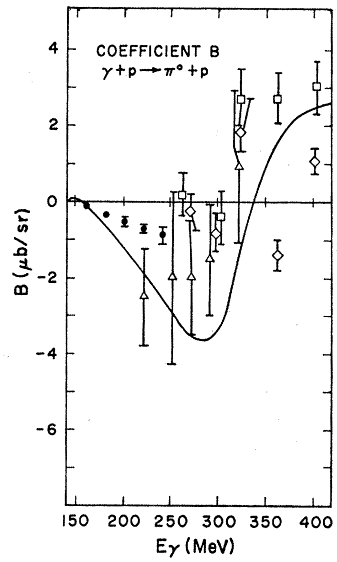

FIG. 9. The coefficient B for the process $\gamma p\rightarrow\pi^0 p$. The notation for the data is the same as in Fig. 8.

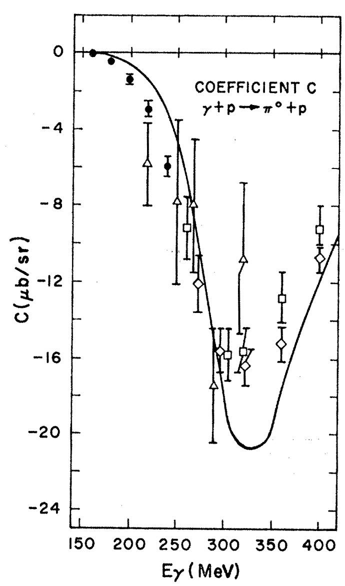

FIG. 10. Coefficient C for the process $\gamma p\rightarrow\pi^0 p$. The notation for the data is the same as in Fig. 8.

that the ratio associated with the magnetic-dipole amplitude $M_{1+\mu}{}^{3/2}$ (half the sum of the solid curves in Fig. 4) is smaller than the corresponding CGLN result by 9% at resonance and decreases relative to the CGLN result as the energy increases and (2) that the charge amplitudes do not vanish at resonance—although the amplitude $E_{1+e}{}^{3/2}$ does have a zero just above resonance.

A secondary result is the generalized effective-range formula (3.13) for the 3-3 phase shift. As we indicated at the end of Sec. III this formula can be used not only as a representation of this phase shift that is valid over a wide range of energies but also as a means for a

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precise detemination of the resonance position and the pion-nucleon coupling constant.

The cross sections shown in Figs. 6–10 were calculated under the assumption that the only important singularities of the photoproduction amplitudes are the Born terms and the 3-3 contributions to the dispersion integrals. Examples of other singularities which may be important in the region of the 3-3 resonance are the contributions to the dispersion integrals of the higher-energy pion-nucleon resonances and effects due to the exchange of ρ mesons. While ρ exchange contributes directly only to the isoscalar (0) photoproduction amplitudes, it also contributes to the isospin $\frac{3}{2}$ scattering amplitude and thus will affect the present results for the 3-3 amplitude ratios. The contribution of the ρ meson to the 3–3 scattering amplitude, which has been treated by Frautschi and Walecka, has an energy dependence and sign consistent with a decrease in the 3-3 amplitude ratios below resonance.[26] This contribution can readily be incorporated into the present expressions for the scattering amplitude and photoproduction amplitude ratios, provided that a redetermination of the parameter ω_m is carried out.

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APPENDIX

The projections for the Born parts of the 3-3 photoproduction amplitude $\mathfrak{F}^{33}$ have been given by Gartenhaus and Blankenbecler[9] in terms of total magnetic moment and charge contributions. The results of these authors may be written in terms of the Born projections for ϕ_1 and ϕ_2, which are given by

$$\phi_{1B}{}^{\mu}=\frac{g(\mu_p-\mu_n)}{kq}\left[\frac{1}{3}Q_{02}\left(\frac{E}{q}\right)-\frac{1}{5}\left(\frac{q}{E+M}\right)Q_{13}\left(\frac{E}{q}\right)\right],$$

$$\phi_{2B}{}^{\mu}=\frac{g(\mu_p-\mu_n)}{(kq)^2}\left[MQ_1\left(\frac{E}{q}\right)-\frac{qM}{E+M}Q_2\left(\frac{E}{q}\right)+\frac{q^2}{5(E+M)}Q_{13}\left(\frac{E}{q}\right)\right],$$

$$\phi_{1B}{}^{e}=\frac{eg}{kq}\left[\frac{1}{3(W+M)}Q_{02}\left(\frac{\omega_q}{q}\right)-\frac{q}{5(E+M)(W-M)}Q_{13}\left(\frac{\omega_q}{q}\right)\right]$$
$$+\frac{eg}{2Mkq}\left[-\frac{(W-M)}{2(W+M)}Q_{02}\left(\frac{E}{q}\right)+\frac{q}{5(E+M)}\frac{(W+M)}{(W-M)}Q_{13}\left(\frac{E}{q}\right)\right], \tag{A1}$$

$$\phi_{2B}{}^{e}=\frac{eg}{k^2}\left[\frac{1}{5(W-M)(E+M)}Q_{13}\left(\frac{\omega_q}{q}\right)\right]-\frac{eg}{2Mk^2}\left[\frac{(W+M)}{5(W-M)(E+M)}Q_{13}\left(\frac{E}{q}\right)\right],$$

$$Q_{mn}=Q_m-Q_n\,,$$

where the Q_m are the Legendre functions of the second kind:

$$Q_0(a)=\tfrac{1}{2}\ln\left(\frac{a+1}{a-1}\right),\quad Q_1(a)=aQ_0(a)-1\quad (m+1)Q_{m+1}(a)=(2m+1)aQ_m(a)-mQ_{m-1}(a)\,,\quad m=1,2,3,\cdots.$$

The Born contribution to the 3-3 scattering amplitude is given by[9]

$$\psi_B=\frac{-g^2}{q^4}\left[(W-M)\alpha(W)-\frac{1}{2}\frac{(E-M)(W+M)\gamma(W)}{E+M}\right], \tag{A2}$$

where g is the unrationalized renormalized pion-nucleon coupling constant and where α and γ are given by

$$\alpha=1-\tfrac{1}{2}a\ln[(a+1)/(a-1)],\quad \gamma=3a+\tfrac{1}{2}(1-3a^2)\ln[(a+1)/(a-1)],\quad a=[E\omega_q-\tfrac{1}{2}]/q^2.$$

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